Faithful real embedding of a three-dimensional complex Kochen-Specker configuration

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We describe a phase-adjusted realification procedure that embeds any finite set of rays in $\mathbb{C}^3$ into $\mathbb{R}^6$. By assigning an appropriate phase to each ray before applying the standard coordinatewise map, we can arrange that two rays are orthogonal in $\mathbb{C}^3$ if and only if their images are orthogonal in $\mathbb{R}^6$, so the construction yields a faithful orthogonal representation of the original complex configuration. As a concrete example, we consider the 165 projectively distinct rays used in a $\mathbb{C}^3$ Kochen-Specker configuration obtained from mutually unbiased bases, list these 165 rays explicitly in $\mathbb{C}^3$, and give for each of them its image in $\mathbb{R}^6$ under the canonical realification map. We also note that, because the original three -element contexts are no longer maximal in $\mathbb{R}^6$, the embedded configuration admits two-valued states even though its realization with maximal contexts in $\mathbb{C}^3$ is Kochen-Specker uncolorable.

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I. A KOCHEN-SPECKER PROOF FOR THE NONEQUIVALENCE OF REAL AND COMPLEX HILBERT SPACE

A foundational question in quantum theory is whether the use of complex numbers is merely a matter of mathematical convenience or a fundamental feature of the physical formalism [1–13]. On the one hand, any finite-dimensional complex Hilbert space $\mathbb{C}^n$ can be represented as a real Hilbert space $\mathbb{R}^{2n}$ via the usual "realification" map that separates real and imaginary parts. This is a typical instance of the general fact that, under sufficiently permissive assumptions (allowing a change of dimension and additional structure such as a complex-structure operator), a real formulation of a complex theory is always possible. Moreover, for many quantum information processing tasks such as quantum computation and Bell nonlocality, standard real quantum theory in this sense is already sufficient to reproduce the operational predictions of the complex theory [9–12]. On the other hand, recent theoretical and experimental work has shown that, if one insists on keeping fixed certain natural structural ingredients (composition via tensor products, the notion of locality, and the usual Hilbert-space interpretation of contexts as maximal orthonormal sets), then there exist correlations and logical structures that are realized in complex Hilbert spaces but cannot be obtained from any purely real Hilbert-space model [1–5,8,13].

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It is therefore useful to distinguish the following three viewpoints. (i) Whether quantum theory can be formulated over the reals depends on which structural assumptions one demands to be preserved. (ii) For a large class of quantum information tasks, real quantum theory is operationally sufficient. (iii) There are also more pragmatic, representation-focused approaches that encode complex quantum mechanics into a real framework at the price of introducing a formal structure different from quantum theory [4,14].

Against this background, in this paper we present a complementary, Kochen-Specker (KS) type argument on the "no-go" side for the logical inequivalence of three-dimensional real and complex Hilbert-space quantum theories, within the standard Hilbert-space framework where sharp yes-no propositions correspond to rays and physical contexts are identified with maximal sets of mutually orthogonal rays. Rather than working with inequalities and probabilities, our approach is based on the logical framework of the Kochen-Specker theorem: we consider a finite set of rays and their orthogonality relations and derive a direct contradiction with the existence of classical truth assignments, formalized by two-valued states. In addition, we exploit a concrete phase-adjusted realification into $\mathbb{R}^6$, which provides an explicit example of the representation-focused approach in viewpoint (iii).

We focus on a specific KS-type configuration introduced by Cabello [14] as a threefold extension of the Yu-Oh configuration [15]. As pointed out by Harding and Salinas Schmeis [16] and analyzed in detail by Navara and Svozil [17], a key structural ingredient of this configuration is the existence, in $\mathbb{C}^3$, of a complete set of mutually unbiased bases (MUBs). Two orthonormal bases are mutually unbiased if every vector of one basis has equal "overlap" with every vector of the other. The theory of MUBs is a rich subject with deep connections to quantum information and finite geometry [18–23]. For the present work, the crucial facts are as follows [22,23].

(i) In $\mathbb{C}^3$, one can construct a complete set of $D+1=4$ MUBs.

(ii) In $\mathbb{R}^3$, it is impossible to find even two mutually unbiased orthonormal bases.

The Yu-Oh-Cabello-KS configuration we consider [14] is built explicitly from the four MUBs in $\mathbb{C}^3$ and consists of 165 rays arranged into 130 contexts; in Sec. III we describe this configuration in detail.

The present paper has two aims. First, building on the work of Harding and Salinas Schmeis [16] and of Navara and Svozil [17], we recast their observations in a MUB-based setting by collecting in one place the explicit 165-ray, 130-context KS configuration in $\mathbb{C}^3$ due to Cabello [14] and explaining, in this concrete example, why its orthogonality relations cannot be realized by any set of unit vectors in $\mathbb{R}^3$ when contexts are required to be maximal. Thus, while neither the configuration nor the impossibility of such a real realization are new, our treatment provides a self-contained finite illustration of a logical structure—a set of propositions and contexts—that exists in three-dimensional complex quantum theory but not in three-dimensional real quantum theory under the same interpretation of contexts.

Second, and this is the genuinely new part of the present work, we show that the same orthogonality hypergraph does admit a faithful orthogonal representation [24] in $\mathbb{R}^6$: by appropriately choosing the phase of each ray before applying the standard coordinatewise separation into real and imaginary parts, we embed all 165 rays as vectors in $\mathbb{R}^6$ so that two rays are orthogonal in $\mathbb{C}^3$ if and only if their images are orthogonal in $\mathbb{R}^6$. However, in this real embedding each three-element context now spans only a three-dimensional subspace of $\mathbb{R}^6$ and is no longer maximal. It can always be extended to a full orthonormal basis of $\mathbb{R}^6$ by adding three extra vectors outside the configuration. This allows one to construct two-valued states by assigning value 0 to all 165 embedded rays and value 1 to a suitable extra basis vector in each completed context.

In this way, the same finite orthogonality hypergraph, realized as maximal contexts in $\mathbb{C}^3$, is KS uncolorable and admits no classical two-valued states, whereas its faithful realization in $\mathbb{R}^6$ does admit classical two-valued states once contexts are interpreted as maximal in the larger space. Our analysis therefore highlights a nuanced picture: at fixed dimension three, and with the usual identification of contexts with maximal orthonormal sets, there exist logical structures realizable in complex but not in real quantum theory, in line with earlier work [16,17]. At the same time, our phase-adjusted realification shows that *orthogonality relations alone* do not separate real quantum theory from complex quantum theory, since any finite orthogonality hypergraph arising from rays in $\mathbb{C}^3$ admits a faithful orthogonal representation in a suitable real Hilbert space. This clarifies the scope of KS-style arguments in the real vs complex debate and complements earlier correlation-based separations between real and complex Hilbert-space models.

Our argument is, in this sense, system-dependent but basis-independent: it singles out a specific three-dimensional configuration and shows that it can be realized in complex Hilbert space but not in real Hilbert space of the same dimension when contexts are required to be maximal, yet the statement does not depend on any preferred basis. This positions our work between basis- and system-dependent, representation-focused approaches (such as resource theories of coherence or imaginarity) and more global, semi-device-independent separations between real and complex quantum theories in the style of Renou *et al.* [1].

II. PHASE-ADJUSTED REALIFICATION IN $\mathbb{C}^3$

A. Setup

Let $\{[\psi_k]\}_{k=1}^N$ be a finite set of rays in $\mathbb{C}^3$, with normalized representatives $\psi_k \in \mathbb{C}^3$, and let

$$c_{k\ell} := \langle \psi_k, \psi_\ell \rangle_{\mathbb{C}} \tag{1}$$

denote their complex inner products.

For each ray we may multiply ψ_k by an arbitrary unit complex phase $e^{i\theta_k}$ without changing the ray. We first define the canonical realification map $\Phi_0 : \mathbb{C}^3 \to \mathbb{R}^6$ as

$$\Phi_0(z_1, z_2, z_3) := (\mathrm{Re} z_1, \mathrm{Re} z_2, \mathrm{Re} z_3, \mathrm{Im} z_1, \mathrm{Im} z_2, \mathrm{Im} z_3). \tag{2}$$

In the physics literature, this invariance under multiplication by a global phase $e^{i\theta_k} \in U(1)$ is often referred to as an (Abelian) gauge freedom. We exploit this freedom and define a *phase-adjusted realification*:

$$R_k := \Phi_0(e^{i\theta_k}\,\psi_k) \in \mathbb{R}^6. \tag{3}$$

Note that the real part of the complex inner product is exactly the real dot product:

$$\begin{aligned} R_k \cdot R_\ell &= \Phi_0(e^{i\theta_k}\psi_k) \cdot \Phi_0(e^{i\theta_\ell}\psi_\ell) \\ &= \mathrm{Re}\langle e^{i\theta_k}\psi_k, e^{i\theta_\ell}\psi_\ell\rangle_{\mathbb{C}} \\ &= \mathrm{Re}(e^{i(\theta_\ell-\theta_k)} c_{k\ell}). \end{aligned} \tag{4}$$

If $c_{k\ell} = 0$, then $R_k \cdot R_\ell = 0$ for all phases, so complex orthogonality is always preserved. The only potential issue is that for $c_{k\ell} \neq 0$ we might accidentally choose phases such that $R_k \cdot R_\ell = 0$, thereby introducing a spurious orthogonality.

B. Forbidden phase differences for nonorthogonal pairs

Assume $c_{k\ell} \neq 0$. Write $c_{k\ell} = |c_{k\ell}| e^{i\varphi_{k\ell}}$. Then Eq. (4) becomes

$$R_k \cdot R_\ell = |c_{k\ell}| \cos[(\theta_\ell - \theta_k) + \varphi_{k\ell}]. \tag{5}$$

Thus $R_k \cdot R_\ell = 0$ if and only if

$$(\theta_\ell - \theta_k) + \varphi_{k\ell} \equiv \frac{\pi}{2} \pmod{\pi}. \tag{6}$$

For a fixed θ_k and fixed nonzero $c_{k\ell}$, this excludes at most two values of θ_ℓ modulo 2π:

$$\theta_\ell \equiv \theta_k - \varphi_{k\ell} + \frac{\pi}{2} \quad \text{or} \quad \theta_\ell \equiv \theta_k - \varphi_{k\ell} + \frac{3\pi}{2} \pmod{2\pi}. \tag{7}$$

Changing θ_k shifts both forbidden values of θ_ℓ by the same amount, reflecting the overall phase (gauge) freedom in the choice of ray representatives.

C. Existence of a faithful embedding

We construct the phases $\theta_1, \ldots, \theta_N$ inductively.

(i) Set $\theta_1 := 0$ arbitrarily.

(ii) Suppose $\theta_1, \ldots, \theta_{m-1}$ have been chosen such that for all nonorthogonal pairs (k, ℓ) with $1 \leqslant k < \ell < m$ we have $R_k \cdot R_\ell \neq 0$.

(iii) For the new ray m, consider all pairs (k, m) with $1 \leqslant k < m$ and $c_{km} \neq 0$. For each such pair, the two forbidden values of θ_m described above define a finite set $F_{k,m} \subset S^1$, where S^1 denotes the unit circle.

Let

$$F_m := \bigcup_{\substack{1 \leqslant k < m \\ c_{km} \neq 0}} F_{k,m}, \tag{8}$$

which is a finite subset of the unit circle. Choose θ_m anywhere on $S^1 \setminus F_m$—that is, from the set of admissible phase values excluding the finite forbidden set F_m. By construction, for every nonorthogonal pair (k, m) we have $R_k \cdot R_m \neq 0$, while pairs with $c_{km} = 0$ remain orthogonal for all phases. Since the forbidden set F_m is finite and the circle is connected, such a θ_m always exists; in fact, all but finitely many phases work. Proceeding by induction on m, we can successively choose the phases θ_m so that no unwanted orthogonality is introduced, obtaining a full set of phases $\{\theta_k\}_{k=1}^N$ such that

$$c_{k\ell} = 0 \iff R_k \cdot R_\ell = 0. \tag{9}$$

Thus, $\{R_k\}_{k=1}^N \subset \mathbb{R}^6$ is a faithful orthogonal representation of the original finite set of rays in $\mathbb{C}^3$.

In particular, if $\{\psi_i, \psi_j, \psi_k\}$ form an orthonormal basis of $\mathbb{C}^3$, then each pair is orthogonal in $\mathbb{C}^3$; hence, each pair R_i, R_j, R_k is orthogonal in $\mathbb{R}^6$, so orthogonal triples (contexts) are preserved.

III. EXPLICIT LIST OF THE 165 RAYS AND THEIR IMAGES IN $\mathbb{R}^6$

A. Complex notation

In the previous section, we showed in general that any finite family of rays in $\mathbb{C}^3$ admits a faithful orthogonal representation in $\mathbb{R}^6$ by an appropriate choice of phases and the realification map Φ_0. We now specialize this construction to the concrete KS-type configuration introduced by Cabello and described in Sec. III: in this section, we fix explicit representatives for the 165 rays and write down their images in $\mathbb{R}^6$.

To keep the expressions compact, it is convenient to exploit the algebraic structure of this configuration. Since it is built from four mutually unbiased bases in dimension three, all vector coordinates lie in the cyclotomic field generated by a primitive third root of unity. We therefore introduce the shorthand

$$\omega := e^{2\pi i/3} = -\frac{1}{2} + i\,\frac{\sqrt{3}}{2}, \quad \omega^2 = \overline{\omega} = -\frac{1}{2} - i\,\frac{\sqrt{3}}{2}, \tag{10}$$

with $1 + \omega + \omega^2 = 0$.

Every coordinate in the 165 rays below is a real multiple of 1, ω, or ω^2, with integer coefficients in $\{-2, -1, 0, 1, 2\}$. For any $z = a + b\,\omega + c\,\omega^2$ ($a, b, c \in \mathbb{R}$), we have

$$\mathrm{Re}z = a - \frac{b + c}{2}, \tag{11}$$

$$\mathrm{Im}z = \frac{\sqrt{3}}{2}\,(b - c). \tag{12}$$

In particular, $0 \mapsto (0, 0)$, and

$$\begin{aligned} 1 &\mapsto (1, 0), & -1 &\mapsto (-1, 0), \\ \omega &\mapsto \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right), & \omega^2 &\mapsto \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right), \\ 2\omega &\mapsto \left(-1, \sqrt{3}\right), & 2\omega^2 &\mapsto \left(-1, -\sqrt{3}\right), \\ -\omega &\mapsto \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right), & -\omega^2 &\mapsto \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \\ -2\omega &\mapsto \left(1, -\sqrt{3}\right), & -2\omega^2 &\mapsto \left(1, \sqrt{3}\right). \end{aligned} \tag{13}$$

Since three complex components are encoded as six real ones, the resulting $\mathbb{R}^6$ representation is not unique: any permutation of the six real coordinates gives an orthogonally equivalent configuration. In particular, there are $3! \times 2^3 = 48$ permutations that merely reshuffle the three complex entries and, for each entry, possibly swap its real and imaginary parts. For example, the vector $(1, 1, 1, 0, 0, 0)$ is permutationally equivalent to (1,0,1,0,1,0) via the permutation $(2\,4\,5\,3)$ on $\{1, \ldots, 6\}$; that is,

$$(x_1, x_2, x_3, x_4, x_5, x_6) \mapsto (x_1, x_4, x_2, x_5, x_3, x_6).$$

More generally, composing our phase-adjusted realification with any real orthogonal transformation of $\mathbb{R}^6$ yields another faithful orthogonal representation of the same 165-ray configuration: all such choices are isomorphic from the point of view of the orthogonality hypergraph and of KS colorability. In Table I, we fix a particularly simple "canonical gauge," namely, the ordering $(\mathrm{Re}z_1, \mathrm{Re}z_2, \mathrm{Re}z_3, \mathrm{Im}z_1, \mathrm{Im}z_2, \mathrm{Im}z_3)$, but none of our conclusions depend on this specific coordinate choice.

B. Explicit realification

For concreteness, in Table I we provide the explicit coordinates for all 165 rays in the configuration. The orthogonality graph of these rays—which visualizes the logical structure of the proof—is depicted in Fig. 2 of Ref. [16], as well as in Fig. 1 of Ref. [14] and Fig. 1 of Ref. [17]. In our tabulation, each row contains a label, the corresponding ray in $\mathbb{C}^3$, and its image in $\mathbb{R}^6$ under the canonical realification map (2) (i.e., with phases $\theta_k = 0$). For a vector $(z_1, z_2, z_3) \in \mathbb{C}^3$, we write its real image as

$$(\mathrm{Re}z_1, \mathrm{Re}z_2, \mathrm{Re}z_3, \mathrm{Im}z_1, \mathrm{Im}z_2, \mathrm{Im}z_3) \in \mathbb{R}^6. \tag{14}$$

The 165 rays listed in Table I are defined only up to overall complex phases. In order to compare their realifications in $\mathbb{R}^6$, it is necessary to make a consistent phase choice. A direct symbolic computation shows that one can assign to each ray $v_k \in \mathbb{C}^3$ a phase $\theta_k = n_k\pi/K$ with integers n_k and some fixed (nonunique) integer K (we may take, for instance, $K = 1009$), such that the following two conditions hold simultaneously: (i) whenever $\langle v_i, v_j\rangle = 0$ in $\mathbb{C}^3$, the corresponding realified vectors $R_i, R_j \in \mathbb{R}^6$ satisfy $R_i \cdot R_j = 0$; and (ii) whenever $\langle v_i, v_j\rangle \neq 0$, the realifications satisfy $R_i \cdot R_j \neq 0$. In particular, this phase choice introduces no spurious orthogonality relations in $\mathbb{R}^6$ and preserves all of the orthogonality relations that define the original KS hypergraph in $\mathbb{C}^3$. We verified the existence of such phases numerically with

TABLE I. All 165 rays in $\mathbb{C}^3$ and their images in $\mathbb{R}^6$ under the canonical realification (2) [i.e., with all phase adjustments $\theta_k = 0$ as defined in Eq. (3)]. This brings about spurious orthogonalities in $\mathbb{R}^6$, such as between u_1 and u_4 that are not present in $\mathbb{C}^3$. A strictly faithful embedding with no spurious orthogonalities requires the additional phase adjustments described in Eq. (7).

Label	$\mathbb{C}^3$ vector	$\mathbb{R}^6$ vector
a_{11}	(1,0,0)	(1, 0, 0, 0, 0, 0)
a_{21}	(0,1,0)	(0, 1, 0, 0, 0, 0)
a_{31}	(0,0,1)	(0, 0, 1, 0, 0, 0)
u_1	(1,1,1)	(1, 1, 1, 0, 0, 0)
u_2	$(1, \omega, \omega^2)$	$\left(1, -\frac{1}{2}, -\frac{1}{2}, 0, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}\right)$
u_3	$(1, \omega^2, \omega)$	$\left(1, -\frac{1}{2}, -\frac{1}{2}, 0, -\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)$
u_4	$(1, \omega, \omega)$	$\left(1, -\frac{1}{2}, -\frac{1}{2}, 0, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)$
u_5	$(1, \omega^2, 1)$	$\left(1, -\frac{1}{2}, 1, 0, -\frac{\sqrt{3}}{2}, 0\right)$
u_6	$(1, 1, \omega^2)$	$\left(1, 1, -\frac{1}{2}, 0, 0, -\frac{\sqrt{3}}{2}\right)$
u_7	$(1, \omega^2, \omega^2)$	$\left(1, -\frac{1}{2}, -\frac{1}{2}, 0, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}\right)$
u_8	$(1, \omega, 1)$	$\left(1, -\frac{1}{2}, 1, 0, \frac{\sqrt{3}}{2}, 0\right)$
u_9	$(1, 1, \omega)$	$\left(1, 1, -\frac{1}{2}, 0, 0, \frac{\sqrt{3}}{2}\right)$
b_{11}	(0,1,1)	(0, 1, 1, 0, 0, 0)
b_{12}	$(0, \omega, \omega^2)$	$\left(0, -\frac{1}{2}, -\frac{1}{2}, 0, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}\right)$
b_{13}	$(0, \omega^2, \omega)$	$\left(0, -\frac{1}{2}, -\frac{1}{2}, 0, -\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)$
b_{21}	(1,0,1)	(1, 0, 1, 0, 0, 0)
b_{22}	$(1, 0, \omega^2)$	$\left(1, 0, -\frac{1}{2}, 0, 0, -\frac{\sqrt{3}}{2}\right)$
b_{23}	$(1, 0, \omega)$	$\left(1, 0, -\frac{1}{2}, 0, 0, \frac{\sqrt{3}}{2}\right)$
b_{31}	(1,1,0)	(1, 1, 0, 0, 0, 0)
b_{32}	$(1, \omega, 0)$	$\left(1, -\frac{1}{2}, 0, 0, \frac{\sqrt{3}}{2}, 0\right)$
b_{33}	$(1, \omega^2, 0)$	$\left(1, -\frac{1}{2}, 0, 0, -\frac{\sqrt{3}}{2}, 0\right)$
c_{11}	$(0, 1, -1)$	$(0, 1, -1, 0, 0, 0)$
c_{12}	$(0, \omega, -\omega^2)$	$\left(0, -\frac{1}{2}, \frac{1}{2}, 0, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)$
c_{13}	$(0, \omega^2, -\omega)$	$\left(0, -\frac{1}{2}, \frac{1}{2}, 0, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}\right)$
c_{21}	$(-1, 0, 1)$	$(-1, 0, 1, 0, 0, 0)$
c_{22}	$(-\omega, 0, 1)$	$\left(\frac{1}{2}, 0, 1, -\frac{\sqrt{3}}{2}, 0, 0\right)$
c_{23}	$(-\omega^2, 0, 1)$	$\left(\frac{1}{2}, 0, 1, \frac{\sqrt{3}}{2}, 0, 0\right)$
c_{31}	$(1, -1, 0)$	$(1, -1, 0, 0, 0, 0)$
c_{32}	$(\omega^2, -1, 0)$	$\left(-\frac{1}{2}, -1, 0, -\frac{\sqrt{3}}{2}, 0, 0\right)$
c_{33}	$(\omega, -1, 0)$	$\left(-\frac{1}{2}, -1, 0, \frac{\sqrt{3}}{2}, 0, 0\right)$
d_{11}	$(-2, 1, 1)$	$(-2, 1, 1, 0, 0, 0)$
d_{12}	$(-2, \omega, \omega^2)$	$\left(-2, -\frac{1}{2}, -\frac{1}{2}, 0, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}\right)$
d_{13}	$(-2, \omega^2, \omega)$	$\left(-2, -\frac{1}{2}, -\frac{1}{2}, 0, -\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)$
d_{14}	$(-2, \omega, \omega)$	$\left(-2, -\frac{1}{2}, -\frac{1}{2}, 0, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)$
d_{15}	$(-2, \omega^2, 1)$	$\left(-2, -\frac{1}{2}, 1, 0, -\frac{\sqrt{3}}{2}, 0\right)$
d_{16}	$(-2, 1, \omega^2)$	$\left(-2, 1, -\frac{1}{2}, 0, 0, -\frac{\sqrt{3}}{2}\right)$
d_{17}	$(-2, \omega^2, \omega^2)$	$\left(-2, -\frac{1}{2}, -\frac{1}{2}, 0, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}\right)$
d_{18}	$(-2, \omega, 1)$	$\left(-2, -\frac{1}{2}, 1, 0, \frac{\sqrt{3}}{2}, 0\right)$
d_{19}	$(-2, 1, \omega)$	$\left(-2, 1, -\frac{1}{2}, 0, 0, \frac{\sqrt{3}}{2}\right)$
d_{21}	$(1, -2, 1)$	$(1, -2, 1, 0, 0, 0)$
d_{22}	$(\omega^2, -2, \omega)$	$\left(-\frac{1}{2}, -2, -\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}\right)$

TABLE I. (*Continued.*)

Label	$\mathbb{C}^3$ vector	$\mathbb{R}^6$ vector
d_{23}	$(\omega, -2, \omega^2)$	$\left(-\frac{1}{2}, -2, -\frac{1}{2}, \frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}\right)$
d_{24}	$(\omega^2, -2, 1)$	$\left(-\frac{1}{2}, -2, 1, -\frac{\sqrt{3}}{2}, 0, 0\right)$
d_{25}	$(\omega, -2, \omega)$	$\left(-\frac{1}{2}, -2, -\frac{1}{2}, \frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}\right)$
d_{26}	$(1, -2, \omega^2)$	$\left(1, -2, -\frac{1}{2}, 0, 0, -\frac{\sqrt{3}}{2}\right)$
d_{27}	$(\omega, -2, 1)$	$\left(-\frac{1}{2}, -2, 1, \frac{\sqrt{3}}{2}, 0, 0\right)$
d_{28}	$(\omega^2, -2, \omega^2)$	$\left(-\frac{1}{2}, -2, -\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}\right)$
d_{29}	$(1, -2, \omega)$	$\left(1, -2, -\frac{1}{2}, 0, 0, \frac{\sqrt{3}}{2}\right)$
d_{31}	$(1, 1, -2)$	$(1, 1, -2, 0, 0, 0)$
d_{32}	$(\omega, \omega^2, -2)$	$\left(-\frac{1}{2}, -\frac{1}{2}, -2, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0\right)$
d_{33}	$(\omega^2, \omega, -2)$	$\left(-\frac{1}{2}, -\frac{1}{2}, -2, -\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0\right)$
d_{34}	$(\omega^2, 1, -2)$	$\left(-\frac{1}{2}, 1, -2, -\frac{\sqrt{3}}{2}, 0, 0\right)$
d_{35}	$(1, \omega^2, -2)$	$\left(1, -\frac{1}{2}, -2, 0, -\frac{\sqrt{3}}{2}, 0\right)$
d_{36}	$(\omega, \omega, -2)$	$\left(-\frac{1}{2}, -\frac{1}{2}, -2, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0\right)$
d_{37}	$(\omega, 1, -2)$	$\left(-\frac{1}{2}, 1, -2, \frac{\sqrt{3}}{2}, 0, 0\right)$
d_{38}	$(1, \omega, -2)$	$\left(1, -\frac{1}{2}, -2, 0, \frac{\sqrt{3}}{2}, 0\right)$
d_{39}	$(\omega^2, \omega^2, -2)$	$\left(-\frac{1}{2}, -\frac{1}{2}, -2, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0\right)$
b_{121}	$(1, 1, -1)$	$(1, 1, -1, 0, 0, 0)$
b_{122}	$(1, \omega, -\omega^2)$	$\left(1, -\frac{1}{2}, \frac{1}{2}, 0, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)$
b_{123}	$(1, \omega^2, -\omega)$	$\left(1, -\frac{1}{2}, \frac{1}{2}, 0, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}\right)$
b_{124}	$(\omega, \omega^2, -\omega^2)$	$\left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)$
b_{125}	$(\omega, 1, -\omega)$	$\left(-\frac{1}{2}, 1, \frac{1}{2}, \frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}\right)$
b_{126}	$(\omega, \omega, -1)$	$\left(-\frac{1}{2}, -\frac{1}{2}, -1, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0\right)$
b_{127}	$(\omega^2, \omega, -\omega)$	$\left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}\right)$
b_{128}	$(\omega^2, 1, -\omega^2)$	$\left(-\frac{1}{2}, 1, \frac{1}{2}, -\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}\right)$
b_{129}	$(\omega^2, \omega^2, -1)$	$\left(-\frac{1}{2}, -\frac{1}{2}, -1, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0\right)$
b_{131}	$(1, -1, 1)$	$(1, -1, 1, 0, 0, 0)$
b_{132}	$(1, -\omega, \omega^2)$	$\left(1, \frac{1}{2}, -\frac{1}{2}, 0, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}\right)$
b_{133}	$(1, -\omega^2, \omega)$	$\left(1, \frac{1}{2}, -\frac{1}{2}, 0, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)$
b_{134}	$(\omega, -\omega^2, \omega^2)$	$\left(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}\right)$
b_{135}	$(\omega, -1, \omega)$	$\left(-\frac{1}{2}, -1, -\frac{1}{2}, \frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}\right)$
b_{136}	$(\omega, -\omega, 1)$	$\left(-\frac{1}{2}, \frac{1}{2}, 1, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0\right)$
b_{137}	$(\omega^2, -\omega, \omega)$	$\left(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)$
b_{138}	$(\omega^2, -1, \omega^2)$	$\left(-\frac{1}{2}, -1, -\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}\right)$
b_{139}	$(\omega^2, -\omega^2, 1)$	$\left(-\frac{1}{2}, \frac{1}{2}, 1, -\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0\right)$
b_{231}	$(-1, 1, 1)$	$(-1, 1, 1, 0, 0, 0)$
b_{232}	$(-1, \omega, \omega^2)$	$\left(-1, -\frac{1}{2}, -\frac{1}{2}, 0, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}\right)$
b_{233}	$(-1, \omega^2, \omega)$	$\left(-1, -\frac{1}{2}, -\frac{1}{2}, 0, -\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)$
b_{234}	$(-\omega, \omega^2, \omega^2)$	$\left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}\right)$
b_{235}	$(-\omega, 1, \omega)$	$\left(\frac{1}{2}, 1, -\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}\right)$
b_{236}	$(-\omega, \omega, 1)$	$\left(\frac{1}{2}, -\frac{1}{2}, 1, -\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0\right)$
b_{237}	$(-\omega^2, \omega, \omega)$	$\left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)$
b_{238}	$(-\omega^2, 1, \omega^2)$	$\left(\frac{1}{2}, 1, -\frac{1}{2}, \frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}\right)$
b_{239}	$(-\omega^2, \omega^2, 1)$	$\left(\frac{1}{2}, -\frac{1}{2}, 1, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0\right)$
e_{11}	(1,2,1)	(1, 2, 1, 0, 0, 0)
e_{12}	$(\omega^2, 2, \omega)$	$\left(-\frac{1}{2}, 2, -\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}\right)$

TABLE I. (*Continued.*)

Label	$\mathbb{C}^3$ vector	$\mathbb{R}^6$ vector
e_{13}	$(\omega, 2, \omega^2)$	$\left(-\frac{1}{2}, 2, -\frac{1}{2}, \frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}\right)$
e_{14}	$(\omega, 2\omega^2, \omega^2)$	$\left(-\frac{1}{2}, -1, -\frac{1}{2}, \frac{\sqrt{3}}{2}, -\sqrt{3}, -\frac{\sqrt{3}}{2}\right)$
e_{15}	$(1, 2\omega^2, 1)$	$(1, -1, 1, 0, -\sqrt{3}, 0)$
e_{16}	$(\omega^2, 2\omega^2, \omega)$	$\left(-\frac{1}{2}, -1, -\frac{1}{2}, -\frac{\sqrt{3}}{2}, -\sqrt{3}, \frac{\sqrt{3}}{2}\right)$
e_{17}	$(\omega^2, 2\omega, \omega)$	$\left(-\frac{1}{2}, -1, -\frac{1}{2}, -\frac{\sqrt{3}}{2}, \sqrt{3}, \frac{\sqrt{3}}{2}\right)$
e_{18}	$(1, 2\omega, 1)$	$(1, -1, 1, 0, \sqrt{3}, 0)$
e_{19}	$(\omega, 2\omega, \omega^2)$	$\left(-\frac{1}{2}, -1, -\frac{1}{2}, \frac{\sqrt{3}}{2}, \sqrt{3}, -\frac{\sqrt{3}}{2}\right)$
e_{21}	$(1,1,2)$	$(1, 1, 2, 0, 0, 0)$
e_{22}	$(\omega, \omega^2, 2)$	$\left(-\frac{1}{2}, -\frac{1}{2}, 2, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0\right)$
e_{23}	$(\omega^2, \omega, 2)$	$\left(-\frac{1}{2}, -\frac{1}{2}, 2, -\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0\right)$
e_{24}	$(\omega, \omega^2, 2\omega^2)$	$\left(-\frac{1}{2}, -\frac{1}{2}, -1, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, -\sqrt{3}\right)$
e_{25}	$(\omega^2, \omega, 2\omega^2)$	$\left(-\frac{1}{2}, -\frac{1}{2}, -1, -\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, -\sqrt{3}\right)$
e_{26}	$(1, 1, 2\omega^2)$	$(1, 1, -1, 0, 0, -\sqrt{3})$
e_{27}	$(\omega^2, \omega, 2\omega)$	$\left(-\frac{1}{2}, -\frac{1}{2}, -1, -\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, \sqrt{3}\right)$
e_{28}	$(\omega, \omega^2, 2\omega)$	$\left(-\frac{1}{2}, -\frac{1}{2}, -1, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, \sqrt{3}\right)$
e_{29}	$(1, 1, 2\omega)$	$(1, 1, -1, 0, 0, \sqrt{3})$
e_{31}	$(2,1,1)$	$(2, 1, 1, 0, 0, 0)$
e_{32}	$(2, \omega, \omega^2)$	$\left(2, -\frac{1}{2}, -\frac{1}{2}, 0, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}\right)$
e_{33}	$(2, \omega^2, \omega)$	$\left(2, -\frac{1}{2}, -\frac{1}{2}, 0, -\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)$
e_{34}	$(2\omega^2, 1, 1)$	$(-1, 1, 1, -\sqrt{3}, 0, 0)$
e_{35}	$(2\omega^2, \omega, \omega^2)$	$\left(-1, -\frac{1}{2}, -\frac{1}{2}, -\sqrt{3}, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}\right)$
e_{36}	$(2\omega^2, \omega^2, \omega)$	$\left(-1, -\frac{1}{2}, -\frac{1}{2}, -\sqrt{3}, -\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)$
e_{37}	$(2\omega, 1, 1)$	$(-1, 1, 1, \sqrt{3}, 0, 0)$
e_{38}	$(2\omega, \omega^2, \omega)$	$\left(-1, -\frac{1}{2}, -\frac{1}{2}, \sqrt{3}, -\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)$
e_{39}	$(2\omega, \omega, \omega^2)$	$\left(-1, -\frac{1}{2}, -\frac{1}{2}, \sqrt{3}, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}\right)$
b_{1121}	$(2, -1, 1)$	$(2, -1, 1, 0, 0, 0)$
b_{1122}	$(2, -\omega, \omega^2)$	$\left(2, \frac{1}{2}, -\frac{1}{2}, 0, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}\right)$
b_{1123}	$(2, -\omega^2, \omega)$	$\left(2, \frac{1}{2}, -\frac{1}{2}, 0, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)$
b_{1124}	$(2, -\omega, \omega)$	$\left(2, \frac{1}{2}, -\frac{1}{2}, 0, -\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)$
b_{1125}	$(2, -\omega^2, 1)$	$\left(2, \frac{1}{2}, 1, 0, \frac{\sqrt{3}}{2}, 0\right)$
b_{1126}	$(2, -1, \omega^2)$	$\left(2, -1, -\frac{1}{2}, 0, 0, -\frac{\sqrt{3}}{2}\right)$
b_{1127}	$(2, -\omega^2, \omega^2)$	$\left(2, \frac{1}{2}, -\frac{1}{2}, 0, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}\right)$
b_{1128}	$(2, -\omega, 1)$	$\left(2, \frac{1}{2}, 1, 0, -\frac{\sqrt{3}}{2}, 0\right)$
b_{1129}	$(2, -1, \omega)$	$\left(2, -1, -\frac{1}{2}, 0, 0, \frac{\sqrt{3}}{2}\right)$
b_{2121}	$(-1, 2, 1)$	$(-1, 2, 1, 0, 0, 0)$
b_{2122}	$(-1, 2\omega, \omega^2)$	$\left(-1, -1, -\frac{1}{2}, 0, \sqrt{3}, -\frac{\sqrt{3}}{2}\right)$
b_{2123}	$(-1, 2\omega^2, \omega)$	$\left(-1, -1, -\frac{1}{2}, 0, -\sqrt{3}, \frac{\sqrt{3}}{2}\right)$
b_{2124}	$(-1, 2\omega, \omega)$	$\left(-1, -1, -\frac{1}{2}, 0, \sqrt{3}, \frac{\sqrt{3}}{2}\right)$
b_{2125}	$(-1, 2\omega^2, 1)$	$(-1, -1, 1, 0, -\sqrt{3}, 0)$
b_{2126}	$(-1, 2, \omega^2)$	$\left(-1, 2, -\frac{1}{2}, 0, 0, -\frac{\sqrt{3}}{2}\right)$
b_{2127}	$(-1, 2\omega^2, \omega^2)$	$\left(-1, -1, -\frac{1}{2}, 0, -\sqrt{3}, -\frac{\sqrt{3}}{2}\right)$
b_{2128}	$(-1, 2\omega, 1)$	$(-1, -1, 1, 0, \sqrt{3}, 0)$
b_{2129}	$(-1, 2, \omega)$	$\left(-1, 2, -\frac{1}{2}, 0, 0, \frac{\sqrt{3}}{2}\right)$
b_{1131}	$(2, 1, -1)$	$(2, 1, -1, 0, 0, 0)$
b_{1132}	$(2, \omega, -\omega^2)$	$\left(2, -\frac{1}{2}, \frac{1}{2}, 0, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)$
b_{1133}	$(2, \omega^2, -\omega)$	$\left(2, -\frac{1}{2}, \frac{1}{2}, 0, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}\right)$
b_{1134}	$(2, \omega, -\omega)$	$\left(2, -\frac{1}{2}, \frac{1}{2}, 0, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}\right)$
b_{1135}	$(2, \omega^2, -1)$	$\left(2, -\frac{1}{2}, -1, 0, -\frac{\sqrt{3}}{2}, 0\right)$
b_{1136}	$(2, 1, -\omega^2)$	$\left(2, 1, \frac{1}{2}, 0, 0, \frac{\sqrt{3}}{2}\right)$
b_{1137}	$(2, \omega^2, -\omega^2)$	$\left(2, -\frac{1}{2}, \frac{1}{2}, 0, -\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)$
b_{1138}	$(2, \omega, -1)$	$\left(2, -\frac{1}{2}, -1, 0, \frac{\sqrt{3}}{2}, 0\right)$
b_{1139}	$(2, 1, -\omega)$	$\left(2, 1, \frac{1}{2}, 0, 0, -\frac{\sqrt{3}}{2}\right)$
b_{3131}	$(-1, 1, 2)$	$(-1, 1, 2, 0, 0, 0)$
b_{3132}	$(-1, \omega, 2\omega^2)$	$\left(-1, -\frac{1}{2}, -1, 0, \frac{\sqrt{3}}{2}, -\sqrt{3}\right)$
b_{3133}	$(-1, \omega^2, 2\omega)$	$\left(-1, -\frac{1}{2}, -1, 0, -\frac{\sqrt{3}}{2}, \sqrt{3}\right)$
b_{3134}	$(-1, \omega, 2\omega)$	$\left(-1, -\frac{1}{2}, -1, 0, \frac{\sqrt{3}}{2}, \sqrt{3}\right)$
b_{3135}	$(-1, \omega^2, 2)$	$\left(-1, -\frac{1}{2}, 2, 0, -\frac{\sqrt{3}}{2}, 0\right)$
b_{3136}	$(-1, 1, 2\omega^2)$	$(-1, 1, -1, 0, 0, -\sqrt{3})$
b_{3137}	$(-1, \omega^2, 2\omega^2)$	$\left(-1, -\frac{1}{2}, -1, 0, -\frac{\sqrt{3}}{2}, -\sqrt{3}\right)$
b_{3138}	$(-1, \omega, 2)$	$\left(-1, -\frac{1}{2}, 2, 0, \frac{\sqrt{3}}{2}, 0\right)$
b_{3139}	$(-1, 1, 2\omega)$	$(-1, 1, -1, 0, 0, \sqrt{3})$
b_{2231}	$(1, 2, -1)$	$(1, 2, -1, 0, 0, 0)$
b_{2232}	$(1, 2\omega, -\omega^2)$	$\left(1, -1, \frac{1}{2}, 0, \sqrt{3}, \frac{\sqrt{3}}{2}\right)$
b_{2233}	$(1, 2\omega^2, -\omega)$	$\left(1, -1, \frac{1}{2}, 0, -\sqrt{3}, -\frac{\sqrt{3}}{2}\right)$
b_{2234}	$(1, 2\omega, -\omega)$	$\left(1, -1, \frac{1}{2}, 0, \sqrt{3}, -\frac{\sqrt{3}}{2}\right)$
b_{2235}	$(1, 2\omega^2, -1)$	$(1, -1, -1, 0, -\sqrt{3}, 0)$
b_{2236}	$(1, 2, -\omega^2)$	$\left(1, 2, \frac{1}{2}, 0, 0, \frac{\sqrt{3}}{2}\right)$
b_{2237}	$(1, 2\omega^2, -\omega^2)$	$\left(1, -1, \frac{1}{2}, 0, -\sqrt{3}, \frac{\sqrt{3}}{2}\right)$
b_{2238}	$(1, 2\omega, -1)$	$(1, -1, -1, 0, \sqrt{3}, 0)$
b_{2239}	$(1, 2, -\omega)$	$\left(1, 2, \frac{1}{2}, 0, 0, -\frac{\sqrt{3}}{2}\right)$
b_{3231}	$(1, -1, 2)$	$(1, -1, 2, 0, 0, 0)$
b_{3232}	$(1, -\omega, 2\omega^2)$	$\left(1, \frac{1}{2}, -1, 0, -\frac{\sqrt{3}}{2}, -\sqrt{3}\right)$
b_{3233}	$(1, -\omega^2, 2\omega)$	$\left(1, \frac{1}{2}, -1, 0, \frac{\sqrt{3}}{2}, \sqrt{3}\right)$
b_{3234}	$(1, -\omega, 2\omega)$	$\left(1, \frac{1}{2}, -1, 0, -\frac{\sqrt{3}}{2}, \sqrt{3}\right)$
b_{3235}	$(1, -\omega^2, 2)$	$\left(1, \frac{1}{2}, 2, 0, \frac{\sqrt{3}}{2}, 0\right)$
b_{3236}	$(1, -1, 2\omega^2)$	$(1, -1, -1, 0, 0, -\sqrt{3})$
b_{3237}	$(1, -\omega^2, 2\omega^2)$	$\left(1, \frac{1}{2}, -1, 0, \frac{\sqrt{3}}{2}, -\sqrt{3}\right)$
b_{3238}	$(1, -\omega, 2)$	$\left(1, \frac{1}{2}, 2, 0, -\frac{\sqrt{3}}{2}, 0\right)$
b_{3239}	$(1, -1, 2\omega)$	$(1, -1, -1, 0, 0, \sqrt{3})$

a simple backtracking script and confirmed that $K = 1009$ suffices. The existence of such a rational phase assignment demonstrates that the entire configuration admits an analytic (real-algebraic) realification without altering its orthogonality structure.

IV. BREAKDOWN OF ADMISSIBILITY CRITERIA IN $\mathbb{R}^6$

Having established the explicit embedding of the 165-ray configuration into $\mathbb{R}^6$, we now turn to its implications for quantum contextuality. This section serves as a concrete illustration of the general fact that faithful real embeddings do not preserve the KS property when contexts are interpreted as maximal in the larger space. We analyze the admissibility criteria for two-valued states on the embedded set and quantify, via numerical optimization, the extent to which the logical contradiction present in $\mathbb{C}^3$ is relaxed in $\mathbb{R}^6$.

The Kochen-Specker contradiction arises from the impossibility of assigning a two-valued state (also called a valuation) $v(P) \in \{0, 1\}$ to every projector P in a set $\mathcal{S}$ such that the assignment satisfies two fundamental admissibility criteria.

(i) Exclusivity: For any pair of mutually orthogonal projectors P_i, P_j, the values must satisfy $v(P_i + P_j) = v(P_i) + v(P_j)$. In particular, if $v(P_i) = 1$, then $v(P_j) = 0$ for all $P_j \perp P_i$.

(ii) Completeness: The assignment must respect the identity observable, $v(\mathbb{I}) = 1$.

In the original complex space $\mathbb{C}^3$, any context $\mathcal{C} = \{P_1, P_2, P_3\}$ consisting of rank-1 projectors onto an orthonormal basis satisfies $\sum_{i=1}^{3} P_i = \mathbb{I}_{\mathbb{C}^3}$. Combining exclusivity and completeness yields the strict sum rule

$$\sum_{i=1}^{3} v(P_i) = v\left(\sum_{i=1}^{3} P_i\right) = v(\mathbb{I}_{\mathbb{C}^3}) = 1. \tag{15}$$

Thus, within every context in $\mathbb{C}^3$, exactly one projector must be assigned the value 1.

However, under the faithful embedding into $\mathbb{R}^6$, we denote the projectors mapped from a single complex context by $\{\Pi_1, \Pi_2, \Pi_3\}$; more generally, we write $\Pi_{i,j}$ for the projector onto the ith ray in the jth context. The triple $\{\Pi_1, \Pi_2, \Pi_3\}$ spans only a three-dimensional subspace. While they remain mutually orthogonal (satisfying exclusivity), they do not sum to the identity of the embedding space:

$$\sum_{i=1}^{3} \Pi_i = \Pi_{\text{sub}} \neq \mathbb{I}_{\mathbb{R}^6}. \tag{16}$$

Completeness applies only to the full space $\mathbb{R}^6$. To invoke it, one must extend the set $\{\Pi_1, \Pi_2, \Pi_3\}$ with orthogonal projectors $\{\Pi_4, \Pi_5, \Pi_6\}$ completing the basis, such that $\sum_{k=1}^{6} \Pi_k = \mathbb{I}_{\mathbb{R}^6}$. The completeness criterion then imposes

$$\sum_{k=1}^{6} v(\Pi_k) = 1. \tag{17}$$

Restricted to the original embedded context, the strict equality of Eq. (15) relaxes to an inequality:

$$0 \leqslant \sum_{i=1}^{3} v(\Pi_i) \leqslant 1. \tag{18}$$

This relaxation permits the assignment $v(\Pi_1) = v(\Pi_2) = v(\Pi_3) = 0$ [provided $v(\Pi_k) = 1$ for some $k \in \{4, 5, 6\}$]. Consequently, one can assign the value 0 to all 165 embedded rays in $\mathbb{R}^6$ simultaneously without violating exclusivity or completeness, thereby avoiding the KS contradiction. In $\mathbb{C}^3$, the same "all-zero" assignment on the 165 rays would yield $\sum_{i=1}^{3} v(\Pi_i) = 0$ in *every* context, thereby violating the completeness requirement $v(\mathbb{I}_{\mathbb{C}^3}) = 1$.

While Eq. (18) allows for the local possibility that a specific embedded context sums to 1 (i.e., retains the "1" within the embedded subspace), the KS obstruction in $\mathbb{C}^3$ manifests globally in $\mathbb{R}^6$ as a strict bound on the total valuation. Considering the complete set of $|\mathcal{C}| = 130$ contexts in the configuration, as analyzed in detail by Navara and Svozil [17], if it were possible to assign values such that every embedded context retained the value 1, the total sum would be exactly 130. However, since this is equivalent to a Kochen-Specker coloring in $\mathbb{C}^3$ (which is impossible), the global sum is strictly bounded:

$$0 \leqslant \sum_{j=1}^{130} \sum_{i=1}^{3} v(\Pi_{i,j}) < 130. \tag{19}$$

The lower bound of 0 corresponds to the "all-zeros" assignment mentioned earlier where the value 1 is placed in the orthogonal complement for every context. The strict upper bound (<130) implies that there is a minimum nonzero deficit. Numerical optimization over all two-valued assignments compatible with exclusivity and completeness confirms that at least two contexts must unavoidably surrender the value 1 to the extra dimensions of $\mathbb{R}^6$ to resolve the logical contradiction. Thus, the final bound is tightened to

$$0 \leqslant \sum_{j=1}^{130} \sum_{i=1}^{3} v(\Pi_{i,j}) \leqslant 128. \tag{20}$$

V. DISCUSSION

We have described a simple phase-adjusted realification procedure that embeds any finite set of rays in $\mathbb{C}^3$ into $\mathbb{R}^6$ in a way that preserves and reflects complex orthogonality. The construction is purely algebraic and uses only the phase freedom inherent in projective rays. Applied to the 165-ray configuration built from MUBs, it yields a faithful orthogonal representation in $\mathbb{R}^6$ (after suitable phase choices), and, by trivial extension, in $\mathbb{R}^7$ and above.

Table I lists all 165 rays explicitly in $\mathbb{C}^3$ together with their canonical realifications in $\mathbb{R}^6$. For applications requiring strict faithfulness (no extra orthogonality among nonorthogonal rays), one additionally picks phases θ_k as in Sec. II before applying Φ_0, which simply rotates each pair $(\mathrm{Re}z_j, \mathrm{Im}z_j)$ in its two-dimensional real subspace.

Whereas, in its original complex realization in $\mathbb{C}^3$, the 165-ray, 130-context configuration is KS uncolorable (there is no two-valued state assigning 0 and 1 such that in each three-element context exactly one projector has value 1 [14]), the same set of rays, when embedded in $\mathbb{R}^6$, does admit classical two-valued states if we keep only the embedded 165 rays and the original three-element contexts.

The reason is that, in $\mathbb{R}^6$, each such triple now spans only a three-dimensional subspace and is no longer a maximal context: it can always be completed to a full orthonormal basis of $\mathbb{R}^6$ by adding three extra vectors not belonging to our configuration. A valuation that assigns the value 0 to all 165 given rays, and places the value 1 on one of these additional basis vectors for each completed context, then satisfies all Kochen-Specker constraints on the restricted set. Because there is a continuum of choices for the orthonormal complements (each three-dimensional subspace has infinitely many orthonormal bases of its orthogonal complement), one can arrange the completions so that no consistency problems arise across different contexts.

This contrast highlights that a faithful orthogonal representation of a KS hypergraph in a higher-dimensional real space does not automatically preserve KS uncolorability: classical two-valued interpretations exist for the embedded configuration in $\mathbb{R}^6$, whereas they are forbidden for its realization as maximal contexts in $\mathbb{C}^3$.

There are two conceptual messages that emerge from this analysis. First, our construction recovers and reformulates, in explicitly MUB-based terms, the central conclusion of Harding and Salinas Schmeis [16]: in three dimensions there exist finite orthogonality configurations of rank-1 projectors that are representable in $\mathbb{C}^3$ but admit no faithful orthogonal representation [24] in $\mathbb{R}^3$. In particular, the triple Yu-Oh construction [14,15], which can be implemented experimentally for a single qutrit (for instance, via generalized multiport interferometers [25]), realizes such a configuration: its hypergraph of orthogonality relations can be realized by unit vectors in $\mathbb{C}^3$ but not by unit vectors in $\mathbb{R}^3$.

Second, the same triple Yu-Oh configuration, when completed to a Kochen-Specker set in $\mathbb{C}^3$, is KS uncolorable: there is no two-valued state (global 0–1 assignment) compatible with all its three-element contexts. Our phase-adjusted embedding shows that this configuration does admit a faithful orthogonal representation in $\mathbb{R}^6$ (and trivially in $\mathbb{R}^D$ for all $D \geqslant 6$). However, in $\mathbb{R}^6$, each three-element context spans only a three-dimensional subspace and is no longer maximal; it can always be extended to a full orthonormal basis of $\mathbb{R}^6$ by adding three extra vectors outside the given set. One can then construct classical two-valued states by assigning the value 0 to all 165 embedded rays and the value 1 to one of the additional basis vectors in each completed context. Because each three-dimensional subspace has a continuum of orthogonal complements, the completions can be chosen consistently across contexts. Thus, the complex realization in $\mathbb{C}^3$ is inherently KS nonclassical, whereas the real realization in $\mathbb{R}^6$ admits classical hidden-variable (two-valued state) models once the contexts are interpreted as maximal in the larger space.

This sharpens and complements earlier tests that distinguish real and complex quantum theory at the level of measurement statistics [1,12,26]: here the separation appears already at the level of the logical structure, in the existence or absence of two-valued states on the same finite orthogonality hypergraph when realized in $\mathbb{C}^3$ versus in $\mathbb{R}^6$.

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DATA AVAILABILITY

No data were created or analyzed in this study.

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