Classical predictions for intertwined quantum observables are contingent and thus inconclusive

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Classical evaluations of configurations of intertwined quantum contexts induce relations, such as true-implies-false, true-implies-true [1], but also nonseparability among the input and output terminals. When combined, these exploitable configurations (aka gadgets) deliver the strongest form of classical value indefiniteness. However, the choice of the respective configuration among all such collections, and thus the relation of its terminals, remains arbitrary and cannot be motivated by some superselection principle inherent to quantum or classical physics.

QUANTUM CLOUDS AS COLLECTIONS OF INTERTWINED CONTEXTS AND THEIR CLASSICAL DOUBLES

Quantum logic, as conceived by von Neumann [2, 3] and Birkhoff [4], is about the formal/theoretical universe of potential empirical observable propositions, as well as the algebraic relations and operations among them. Every single one of these observables is considered operational "subject to the experimenter's grace" as its actual measurement reflects the experimenter's (subjective) choice to indeed measure one of these potential observables, *versus* its (often continuity of) complementary ones. (This choice is mostly supposed to be *"ex machina"*; that is, outside of the reach of quantum theory, and not subject to nesting [5–7].) Thereby, all the other, then counterfactual, observables remain in a "dormant/hypothetical" realm, an idealist [8, 9] "limbo" of sorts.

Even explorations allowing logical operations exclusively among simultaneously commeasurable observables [10, 11] and permitting partial value definiteness [12, 13] (in the recursion-theoretic sense of Kleene [14]) rely upon, and are thus valid relative to, such collections of counterfactuals. Thereby the predictions/forecasts derived not for a single such collection of observables – here sometimes referred to as cloud or gadget – but for (finite) selections from the multitude of conceivable (finite) collections of observables may be inconsistent.

One way to conceptualize the (nonclassical) performance of quantized systems is in terms of (black) boxes with input and output terminals as interfaces [1]. Like zero-knowledge proofs [15] (a topic the late Specker became interested in) they are supposed to certify that they act "truly quantum mechanically" while at the same time not allowing any inspection (e.g., duplication or opening) other than their performance at the input-output terminals. Fulfillment/certification is usually obtained by the exhibition of certain features or signals usually not encountered by classical devices, among them complementarity and classical value indefiniteness (mostly pronounced as contextuality). For a great variety of such criteria see Table I, as well as the references cited therein, later.

However, the signals obtained from these boxes are far from plain. Indeed in what follows we shall argue that, depending on which hypothetical configurations of (necessarily complementary) "intrinsic" observables are considered, any individual outcome can *ad hoc* be classically (re)interpreted as an indication of nonclassicality. Yet the same outcome could also be in full conformity with a classical interpretation. A combination of such classical models allows any statistical prediction at the terminals. Moreover, there does not seem to exist any convincing reason to choose one of such configurations over another, thus giving rise to either contradictory or arbitrary *ad hoc* signal analysis.

Already Specker [16] contemplated about generalized exotic behaviors even beyond quantum boxes, whereby his criteria for "weirdness" were inspired by scholastic counterfactuals (aka *Infuturabilien*). And the benign outcome of his fable was only made possible by the unmarried daughter's determined alas futile attempt to open the "wrong" – according to her father's strategy – box; at which point he gave in, and marriage ensued. Quantum boxes and as will be later argued quantum clouds are not dissimilar: because of complementarity and classical value indefiniteness (aka contextuality) complete knowledge of the situation is impossible by any known physical means. An immediate idealistic [8, 9, 17] objection to the use of counterfactuals could be that the presupposition of the sort of omni-realism required for a classical analysis of quantum boxes cannot be operational [18] and supported by quantum mechanics. Indeed, the partial algebra approach of Kochen and Specker [10, 11, 19] disallows operations among complementary observables whilst making heavy use of intertwined collections of complementary maximal operators (aka contexts).

However, even classical models based on set representable partition logics [20] such as Moore's initial state identification problem [21] and also Wright's generalized urn model [22, 23] mimic quantum complementarity to a certain degree – indeed, formally up to quantum logics with separable sets of two-valued states [19, Theorem 0, p. 67]. Thereby nonseparability of quantum observables with respect to the set of two-valued states (interpretable as classical truth assignments) serves as a strict criterion for nonclassicality, and also as a criterion against realizations by set-theoretical representable partition logics, even if such two-valued states exist.

In what follows we shall further exploit counterfactual configurations of contexts which are intertwined (this terminology is borrowed from Gleason [24]) in one or more common observable(s). Such counterfactual configurations of contexts will be called *clouds*.

Graph theoretically [25, Appendix] a context can be associated with a complete graph (aka clique). Its vertices are identified with the elements of the context. Adjacency is characterized by comeasurable exclusivity. Clouds are represented by collections of complete graphs (aka clicks or contexts) intertwining at the respective vertices, thereby leaving the edges unchanged.

In quantum mechanics, contexts are identified with orthonormal bases, or equivalently with the maximal operators [26, § 84, Theorem 1, p. 171] which can be (nonuniquely) formed by nondegenerate sums containing all the one-dimensional orthogonal projection operators associated with those respective bases. Elementary propositions are formalized by vectors of these bases of d-dimensional Hilbert space, or by the orthogonal projection operators associated with such vectors [4]. Graph theoretically the vertices are represented by the basis vectors, and adjacency stands for orthogonality of these vectors; that is, the edges represent the (pairwise) orthogonality relations between the vectors (vertices). (Each vertex must be connected to all the other $d-1$ vertices in the respective context by an edge.) Thereby the graph representing a cloud has a faithful orthogonal representation [27, 28] in terms of the elements of the bases representing the respective contexts. The inverse problem of finding some faithful orthogonal representation of a given graph is still open. A necessary condition for the existence of intertwines is that the dimensionality of the vector space is higher than two because in fewer dimensions than three contexts are either identical or disjoint.

Orthogonality hypergraphs [29] are compact representations of clouds which reveal their structural constituents by signifying contexts/cliques/bases: every complete graph K_d is replaced by a *single smooth curve* (usually a straight line) containing distinguished points that represent the vertices. Thereby, the $d(d-1)/2$ edges of any such complete graph K_d are replaced with a single smooth curve. All the vertices "within" this smooth curve represent the mutually orthogonal vectors forming a d-dimensional basis.

Clouds may have various model realizations and representations: a particular cloud may have

(i) a quantum mechanical realization in terms of intertwining orthonormal bases, as mentioned earlier;

(ii) a pseudo-classical realization in terms of partition logic which in turn have automaton logic or generalized urn models;

(iii) a classical realization if there is only a single context involved;

(iv) none of the above (such as a tightly interlinked "triangle" configuration of three contexts with two vertices per context).

Suffice it to say that (i) does not imply (ii), and vice versa. Case (iii) can be interpreted as a subalgebra of all the other groups enumerated, as the cases (i) and (ii) are pastings of contexts or (Boolean) blocks [30].

ENFORCING CLASSICAL TWO-VALUED STATES

The commonly used method for exploring nonclassicality is to consider configurations of type (i) with a quantum realization, upon which a classical interpretation, if it exists, is "enforced" in terms of uniform classical truth and falsity allocations of the associated propositions. Such value assignments can be formalized by two-valued states $s \in \{0,1\}$ or (classical truth) value assignments which are additive and add up to one whenever the propositions are exclusive and within a single context.

The physical intuition behind this formalization is the observation that any d-dimensional context or maximal observable can be interpreted as an array of detectors after a d-port beam splitter [31]. In an ideal experiment, only one detector clicks (associated with the proposition that the system is in the respective state), whereas all the other $d-1$ detectors remain silent.

Such uniform classical interpretations are supposed to be context-independent; that is, the value on intertwining observables which are common to two or more contexts is independent of the context. Besides context-independence of truth assignments at the intertwining observables various variants of such measures assume conditions of increasing strength:

(I) The "measures" or value assignments employed in so-called "contextuality inequalities" merely assume that every proposition is either true or false, regardless of the other propositions in that context which are simultaneously measurable [32]. This allows all possible 2^d possibilities of value assignments in a d-dimensional context with d vertices, thereby vastly expanding the multitude of possible value assignments. With this expansion, all Kochen-Specker sets trivially allow value assignments.

(II) The prevalent assumption of two-valued states or value assignments, also used by Kochen and Specker [19] as well as Pitowsky [33], is that only a *single* one of the d vertices within a d-dimensional context is true, and all the others are false; therefore any isolated d-dimensional context can have only d such standard two-valued value assignments.

(III) An even more restricted rule of value assignment abandons uniform definiteness and supposes [12, 13, 34] that, if all $d-1$ but one vertex in a d dimensional context are false, the remaining one is true, and if one vertex within a d-dimensional context is true, all remaining $d-1$ vertices are false. This latter value assignments allow for *partial* functions which can be undefined.

Type (III) implies type (II) which in turn implies type (I) value assignments.

A set S of two-valued states on a graph G is [35, 36]:

(u) *unital*, if for every $x \in G$ there is a two-valued state $s \in S$ such that $s(x) = 1$;

(s) *separating*, if for every distinct pair of vertices $x, y \in G$ with $x \neq y$ there is an $s \in S$ such that $s(x) \neq s(y)$;

(f) *full*, if for every nonadjacent pair of vertices $x, y \in G$ there is an $s \in S$ such that $s(x) = s(y) = 1$.

A full set of two-valued states is separating, and a separating set of two-valued states is unital. As will be detailed later TIFS/10-gadgets have a nonfull set so two-valued states in the sense of (f). Nonseparability in the sense of (s) indicates nonclassicality. And nonunitality in the sense of (u) discredits the classical predictions of quantum clouds even to a greater degree, probably only challenged by a complete absence of two-valued states.

CHROMATIC SEPARABILITY

As already discussed by Kochen and Specker [19, Theorem 0] nonseparability of (at least one) pair of nonadjacent vertices with respect to the set of two-valued states (interpretable as classical truth assignments) of a graph is arguably the most important signature of nonclassicality. It may be true even if there is an "abundance" of two-valued states. Nonseparability can also be expressed in terms of graph coloring.

A proper (vertex) coloring [25, Appendix] of a graph is a function c from the vertex set to a finite set of "colors" (the positive integers will do) such that, whenever x and y are adjacent vertices, $c(x) \neq c(y)$. The chromatic number of a graph is the least positive integer t such that the graph has a coloring with t colors.

A two-valued state on a (hyper)graph composed/pasted from contexts/cliques, all having an identical number of vertices/clique numbers d can be obtained by "projecting/reducing" colors if the chromatic number of that graph equals d. In this case, in order to obtain a two-valued state, take any proper (vertex) coloring c with d colors and map $d-1$ colors into (the "new color") 0 and one color into (the "new color") 1. Just as the graph coloring c itself such mappings need not to be unique.

Note that the chromatic number of a complete graph must be equal to the clique number because type (II) & (III) two-valued states require that every context/clique must have exactly one vertex with value assignment 1 (and 0 for all the other vertices). (One might conjecture that the set of two-valued states induces graph colorings, in much the same way as it induces a partition logic [20].) At the same time, the clique number renders a bound from below on the chromatic number. Thus if the chromatic number of a graph exceeds the clique number no such two-valued states exist – the "phenomenological consequence" is that, for at least one context/clique, the color projection/reduction is constant – namely 0 – on all vertices of that context/clique.

A coloring is *chromatically separating* two nonadjacent distinct vertices x and y in the vertex set of a graph if there exists a proper vertex coloring such that $c(x) \neq c(y)$. A set of colorings of a particular graph is said to be *separating* if, for any pair of distinct vertices it contains at least one coloring which separates those vertices. The *separable chromatic number* of a graph is the least positive integer t such that the graph has a separating set of colorings with at most t colors.

If the separable chromatic number is higher than the chromatic number of a given graph, then there exist nonadjacent vertices which cannot be "resolved" by any proper graph coloring, or, for that matter, by any derived projected/reduced two-valued state. As a consequence, the graph has no set-theoretic realization as a partition logic, although its chromatic number is the clique number, and there still may exist an abundance of proper colorings and two-valued states [19, Γ_3, p. 70].

It would be interesting to translate (non)unitality and (non)fullness into (sets of) graph coloring. For reasons of brevity, we shall not discuss this here.

FORMATION OF GADGETS AS USEFUL SUBGRAPHS FOR THE CONSTRUCTION OF CLOUDS

The commonly used method seeks cloud configurations with "exotic" classical interpretations. Again, exoticism may express itself in various forms or types. One way is in terms of violations on bounds on classical *conditions of possible experience* [37, p. 229], such as Bell-type inequalities derivable from taking all [type (II), and type (I) for inequalities using only the assumption of noncontextuality [32]] value assignments, forming a correlation polytope by encoding those states into vertices, and solving the hull problem thereof [38–45]. Another, stronger [46] form of nonclassicality is the nonexistence of any such classical interpretation in terms of a type (II) valued assignments [16, 19, 24, 33, 47]; or at least their nonseparability [19, Theorem 0, p. 67].

The explicit construction of such exotic classical interpretations often proceeds by the (successive) application of exploitable subconfigurations of contexts – in graph theoretical terms *gadgets* [48–50] defined as "useful subgraphs". Thereby, gadgets are formed from constituent lower order gadgets of ever-increasing size and functional performance (see also [51, Chapter 12]):

1. 0th order gadget: a single context (aka clique/block/Boolean (sub)algebra/maximal observable/orthonormal basis). This can be perceived as the most elementary form of a true-implies-false (TIFS/10) [1]/01-(maybe better 10)-gadget [50, 52] configuration, because a truth/ value 1 assignment of one of the vertices implies falsity/ value 0 assignments of all the others;

2. 1st order "firefly" gadget: two contexts connected in a single intertwining vertex;

3. 2nd order gadget: two 1st order firefly gadgets connected in a single intertwining vertex;

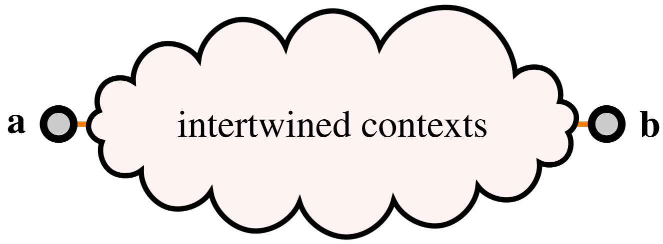

FIG. 1. A collection of possible connections of counterfactuals organised in intertwining contexts and joining **a** and **b**, depicted as a cloud $C(\mathbf{a},\mathbf{b})$.

4. 3rd order house/pentagon/pentagram gadget: one firefly and one 2nd order gadget connected in two intertwining vertices to form a cyclic orthogonality hypergraph;

5. 4rth order 10-gadget: e.g., a Specker bug [1] consisting of two pentagon gadgets connected by an entire context; as well as extensions thereof to arbitrary angles for the terminal ("extreme") points [13, 50];

6. 5th order true-implies-true (TITS) [1]/11-gadget [52]: e.g., Kochen and Specker's Γ_1 [19], consisting of one 10-gadget and one firefly gadget, connected at the respective terminal points (cf. Fig. 6);

That is, gadgets are subconfigurations of clouds. And clouds can be interpreted as gadgets for the composition of bigger clouds.

For the sake of arguing for an idealistic [8, 9, 17] and against a realistic usage of quantum clouds, configurations of intertwined contexts with two fixed propositions as "start" and "end" points **a** and **b** will be studied; as well as methods for constructing such configurations with particular *relational* properties. Whenever there is no preferred, less so unique, path connecting **a** and **b**, all such connections should be treated on an equal basis. We shall call any such collection of counterfactual connections "*clouds* connecting **a** and **b**", denoted by $C(\mathbf{a},\mathbf{b})$, and depict it with a cloud shape symbol, as drawn in Figure 1. (This can in principle be generalized to more than two terminal points.)

Thereby, as the endpoints **a** and **b** remain fixed, one can ask what kind of (classical) *relational information* can be inferred from such two-point quantum clouds. As it turns out, for fixed **a** and **b** quantum clouds can be found which realize a wide variety of conceivable relational properties between **a** and **b**. Table I enumerates these relations.

QUANTUM CLOUDS ENFORCING PARTICULAR FEATURES WHEN INTERPRETED CLASSICALLY

For quantum mechanics, **a** and **b** can be formalized by the two one dimensional projection operators $\mathsf{E}_\mathbf{a} = |\mathbf{a}\rangle\langle\mathbf{a}|$

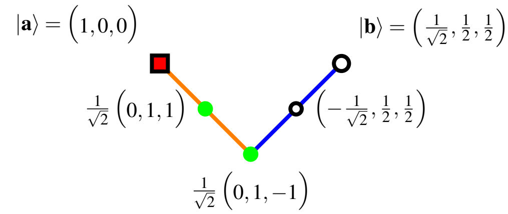

FIG. 2. Orthogonality hypergraph of a cloud consisting of a firefly logic L_{12} connecting **a** and **b**, such that, for type (II) value assignments, **a** true-implies-**b** whatever (quantum 50:50). Truth is encoded by a filled red square, classical falsity by a filled green circle, and arbitrary truth values by circles. [Type (III) value assignments are partial and thus undefined.] L_{12} consists of 5 vertices in just 2 interetwined blocks allowing a separating set of 5 two-valued states and therefore is set representable by partition logics.

and $\mathsf{E}_\mathbf{b} = |\mathbf{b}\rangle\langle\mathbf{b}|$, respectively. For the sake of demonstration we shall study configurations in which $|\mathbf{a}\rangle = \left(1,0,0\right)$ and $|\mathbf{b}\rangle = \left(\frac{1}{\sqrt{2}},\frac{1}{2},\frac{1}{2}\right)$, that is, the quantum prediction yields a probability $|\langle\mathbf{b}|\mathbf{a}\rangle|^2 = \frac{1}{2}$ to find the quantum in a state $|\mathbf{b}\rangle$ if it has been prepared in a state $|\mathbf{a}\rangle$. This configuration can be extended to endpoints with (noncollinear and nonorthogonal) arbitrary relative location by the techniques introduced in Refs. [13, 50].

(a) A quantum cloud configuration for which classical value assignments allow **b** to be either true or false if **a** is true is the firefly configuration [53, pp. 21, 22], depicted in Fig. 2, with five classical value assignments of type (II) [68].

(b) Already Kochen and Specker utilized quantum clouds enforcing classical **a** true-implies-**b** false predictions and their compositions in the construction of a configuration that does not allow a uniform truth assignment [of type (II)]. Stairs [54, p. 588-589] has pointed out that the Specker bug [10, Fig. 1, p. 182] is a quantum cloud configuration which classically enforces **a** true-implies-**b** false: if a quantum system is prepared in such a way that **a** is true – that is, if it is in the state $\mathsf{E}_\mathbf{a}$ – and measured along $\mathsf{E}_\mathbf{b}$, and $|\mathbf{a}\rangle$ and $|\mathbf{b}\rangle$ are not orthogonal or collinear, then any observation of **b** given **a** amounts to a probabilistic proof of nonclassicality: because although quantum probabilities do not vanish, classical value assignments predict that **b** never occurs. Minimal quantum cloud configurations for classical **a** true-implies-**b** false, as well as **a** true-implies-**b** true value assignments [of type (II)] can be found in [1].

As Cabello has pointed out [69, 70], the original Specker bug configuration cannot go beyond the quantum prediction probability threshold $|\langle\mathbf{b}|\mathbf{a}\rangle|^2 = 3^{-2}$ because the angle between **a** and **b** cannot be smaller than $\arccos\frac{1}{3} \approx 1.23096$ radians (71.5°). A configura-

if **a** is true classical value assignments	anectodal, historic quantum realisation	reference to utility or relational properties
imply **b** is independent (arbitrary)	firefly logic L_{12} [53, pp. 21, 22]	
imply **b** false (TIFS/10)	Specker bug logic [10, Fig. 1, p. 182]	[54, p. 588-589], [55], [1]
imply **b** true (TITS)	extended Specker bug logic	[19, Γ_1, p. 68], [56, Sects. II,III, Fig. 1], [57, Fig. C.l. p. 67], [58, p. 394], [59–61], [62–67], [1]
iff **b** true (nonseparability)	combo of intertwined Specker bugs	[19, Γ_3, p. 70]
imply value indefiniteness of **b**	depending on Type (II), (III) assignments	[33], [13]

TABLE I. Some (incomplete) history of the relational properties realizable by two-point quantum clouds.

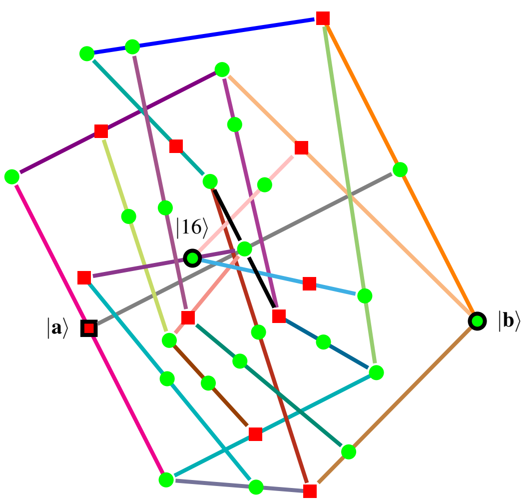

FIG. 3. Orthogonality hypergraph of a nonfull/TIFS/10 cloud even for type (III) value assignments. A faithful orthogonal realization is enumerated in Ref. [13, Table. 1, p. 102201-7]. It consists of 38 vertices in 24 interetwined blocks, endowed with a nonseparating set of 13 two-valued states and therefore is not set representable by partition logics. The state depicted is the only one allowing **a** to be 1. Moreover, this cloud has no unital set of two-valued states as for all of them the vertex represented by the vector $|16\rangle = \frac{1}{\sqrt{10}}\left(2\sqrt{2},1,-1\right)$ and drawn as a solid black circle (and the associated observable) needs to be zero at all classical instantiations.

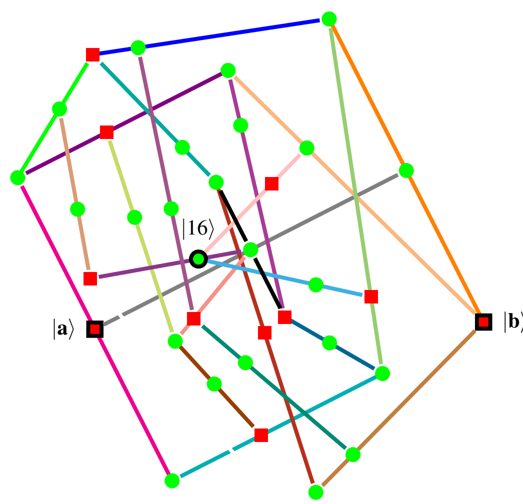

FIG. 4. Orthogonality hypergraph of a TITS/11 cloud even for type (III) value assignments. A faithful orthogonal realization is enumerated in Ref. [13, Table. 1, p. 102201-7]. It consists of 38 vertices in 24 interetwined blocks, endowed with a nonseparating set of 13 two-valued states and therefore is not set representable by partition logics. The state depicted is the only one allowing **a** to be 1. Moreover, this cloud has no unital set of two-valued states as for all of them the vertex represented by the vector $|16\rangle = \frac{1}{\sqrt{10}}\left(2\sqrt{2},1,-1\right)$ and drawn as a solid black circle (and the associated observable) needs to be zero at all classical instantiations.

tion [71, Fig. 5(a)] allowing type (III) TIFS truth assignments with "maximally unbiased" quantum prediction probability $|\langle\mathbf{b}|\mathbf{a}\rangle|^2 = \frac{1}{2}$ is a sublogic of a quantum logic whose realization is enumerated in Ref. [13, Table. 1, p. 102201-7]. It is depicted in Fig. 3. A proof of Theorem 2 in Ref. [50] contains an explicit parametrization of a single TIFS/10 cloud allowing the full range of angles $0 < \angle\mathbf{a},\mathbf{b} < \pi$.

(c) Clifton (note added in proof to Stairs [54, p. 588-589]) presented a **a** true-implies-**b** true (TITS) cloud [56, 72, 73, Sects. II,III, Fig. 1] inspired by Bell [57, Fig. C.l. p. 67] (cf. also Pitowsky [58, p. 394]), as well as by the Specker bug logic [56, Sects. IV, Fig. 2]. Hardy [59–61] as well as Cabello, among others [50, 63–67, 69, 70] utilized similar scenarios for the demonstration of nonclassicality [74, Chapter 14]. Fig. 4 depicts a 11-gadget [71, Fig. 5(b)] with identical endpoints as the 10-gadget discussed earlier and depicted in Fig. 3.

(d) Various parallel and serial compositions of 10- and 11-gadgets serve as a "gadget toolbox" to obtain clouds which, if they are interpreted classically, exhibit other interesting relational properties. For instance, the parallel composition (pasting) of two quantum clouds of the 10-gadget type: one 10-gadget classically demanding **a** true-implies-**b** false and the other 10-gadget classically demanding **b** true-implies-**a** false, results in a quantum cloud which has two observables **a** and **b** which are classically always “opposite”: if one is true, the other one is false, and *vice versa*.

(e) The parallel composition (pasting) of two quantum clouds of the TITS type, with one TITS, classically demanding **a** true-implies-**b** true and the other TITS classically demanding **b** true-implies-**a** true, results in a quantum cloud which has two observables **a** and **b** which are classically nonseparable, which is a sufficient criterion for nonclassicality [19, Theorem 0, p. 67]. As pointed out by Portillo [75] this is equivalent to **a** is true if and only if **b** is true (TIFFTS). Fig. 5 depicts a historic example of such a construction. The serial composition of suitable TITS of the form $\mathbf{a}_1$ true-implies-$\mathbf{a}_2 \cdots \mathbf{a}_{i-1}$ true-implies-$\mathbf{a_i}$ true eventually yields two or more vectors $\mathbf{a}_1$ and $\mathbf{a_i}$ which are mutually orthogonal; a technique employed by Kochen and Specker for the construction of a quantum cloud admitting no type (II) truth assignment [19, Γ_2, p. 69].

(f) The parallel composition (pasting) of the two quantum clouds which respectively represent a 10-gadget and an 11-gadget and identical endpoints **a** and **b** yields a **a** true-implies-**b** value indefinite cloud discussed in Ref. [13].

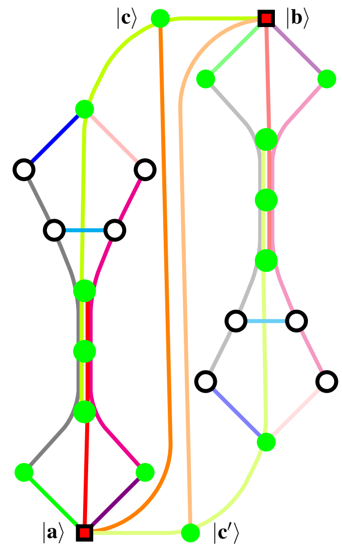

FIG. 5. Orthogonality hypergraph of a TIFFTS cloud for type (II) value assignments, based on a minimal 11-gadgets introduced in Ref. [1, Fig. 6] for dimensions greater than 2. In three dimensions, (i) the three orthogonal “middle” vertices intertwining four contexts vanish, (ii) the two vertices $|\mathbf{c}\rangle$ and $|\mathbf{c}'\rangle$ coincide, and (iii) the two edges connecting $|\mathbf{c}\rangle$ with $|\mathbf{a}\rangle$ and $|\mathbf{c}'\rangle$ with $|\mathbf{b}\rangle$ vanish, rendering the original Specker bug combo introduced by Kochen and Specker [19, Γ_3, p. 70]. Unlike the earlier configurations, this cloud does not allow 50:50 quantum probabilities. Because of nonseparability of its set of two-valued states and its separable chromatic number higher than the clique number it does not allow a set representation by partition logics.

SOME TECHNICAL ISSUES OF GADGET CONSTRUCTION

The concatenation of intertwining gadgets needs to allow a proper faithful orthogonal representation of the resulting compound (hyper)graph while at the same time preserving the structure of these gadgets. Thereby the faithful orthogonal representations of the constituent gadgets cannot always be transferred easily to a faithful orthogonal representation of the resulting compound (hyper)graph.

Suppose, for the sake of a counterexample involving duplicity of vertices after concatenations of gadgets, one would attempt to construct a $G\left(\left(1,0,0\right),\left(0,1,1\right)\right)$ 11 cloud (which would constitute a Kochen-Specker proof as the respective terminal points are orthogonal) by concatenating two 11-gadgets $G\left(\left(1,0,0\right),\left(\frac{1}{\sqrt{2}},\frac{1}{2},\frac{1}{2}\right)\right)$ and $G\left(\left(\frac{1}{\sqrt{2}},\frac{1}{2},\frac{1}{2}\right),\left(0,1,1\right)\right)$ of the type depicted in Fig. 4 by simply rotating all coordinates of the first gadget $\frac{\pi}{4}$ radians (45°) about the axis formed by $\mathbf{b}-\mathbf{a}$. Unfortunately, a straightforward calculation shows that these two 11-gadgets, with the faithful orthogonal realization taken from [13, Table I, p. 102201-7], do not only have the vertex $\left(\frac{1}{\sqrt{2}},\frac{1}{2},\frac{1}{2}\right)$ in common as per construction, but also the three additional vertices $\left(0,\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}}\right)$ and $\left(1,0,0\right)$.

Also, gadgets may not be able to perform as desired. For instance, a standard construction in three dimensions, already used by Kochen and Specker [19, Lemma 1, Γ_1, p. 68] for their construction of a 11-gadget Γ_1 from a Specker bug-type 10-gadget introduced earlier [10, Fig. 1, p. 182], is to take the terminal points **a** and **b** of some TIFS/10 cloud and form the normal vector $\mathbf{c}=\mathbf{a}\times\mathbf{b}$. In a second step, the vector

$$\begin{aligned}\mathbf{d}=\mathbf{b}\times\mathbf{c}=\mathbf{b}\times(\mathbf{a}\times\mathbf{b})\\ =\mathbf{b}^2\mathbf{a}-(\mathbf{a}\cdot\mathbf{b})\,\mathbf{b}=\mathbf{a}-(\mathbf{a}\cdot\mathbf{b})\,\mathbf{b}=\mathbf{a}-\cos\left(\angle\mathbf{a},\mathbf{b}\right)\mathbf{b}\end{aligned} \tag{1}$$

orthogonal to both **b** and **c** is formed. If **a** is true/1 then **b** (because of the 01-gadget) as well as **c** (because of orthogonality with **a**) must be false/0. Therefore **d** must be true, since it completes the context $\{\mathbf{b},\mathbf{c},\mathbf{d}\}$. The situation is depicted in Fig. 6. If all goes well the new cloud $C(\mathbf{a},\mathbf{d})$ is of the TITS/11 type. This is not the case if one uses the TIFS/10-gadget depicted in Fig. 3, as the vector $\mathbf{c}=\left(0,-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ and the new terminal vector $\mathbf{d}=\left(\frac{1}{\sqrt{2}},-\frac{1}{2},-\frac{1}{2}\right)$ also appear in the original TIFS/10-gadget.

For very similar reasons (degeneracy or division through

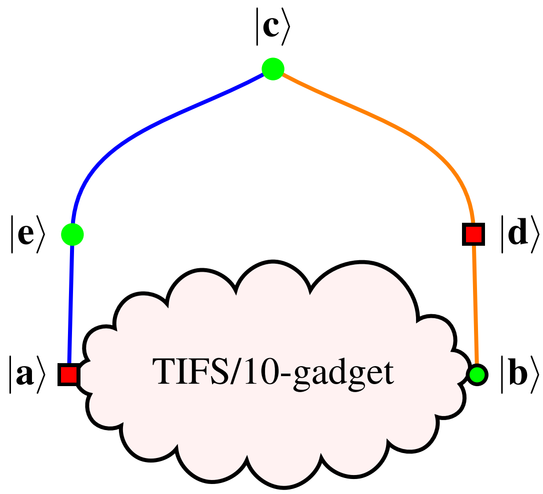

FIG. 6. Standard construction used by Kochen and Specker [19, Lemma 1, Γ_1, p. 68] for obtaining a 11 cloud $C(\mathbf{a},\mathbf{d})$ [or, because of symmetry, $C(\mathbf{b},\mathbf{e})$] from a nonfull/TIFS/10-gadget $C(\mathbf{a},\mathbf{b})$, involving two additional contexts $\{\mathbf{b},\mathbf{d},\mathbf{c}\}$ and $\{\mathbf{a},\mathbf{e},\mathbf{c}\}$.

zero) the 10-gadget introduced in the proof of Theorem 3 in Ref. [50] and depicted in Fig. 7 cannot be extended to an 11 cloud whose end terminals are the "maximal" angle $\frac{\pi}{4}$ radians (45°) apart. For all other allowed angles an extension of this earlier construction of a TIFS/10-gadget to a TITS/11-cloud depicted in Fig. 6 with (without loss of generality and for $0 < \angle \mathbf{a},\mathbf{b} \leq \frac{\pi}{4}$) $\mathbf{a} = \begin{pmatrix}1,0,0\end{pmatrix}$ and $\mathbf{b} = \frac{1}{\sqrt{1+x^2}}\begin{pmatrix}x,1,0\end{pmatrix}$ yields the new terminal vector of the TITS/10-cloud $\mathbf{d} = \frac{1}{\sqrt{1+x^2}}\begin{pmatrix}1,-x,0\end{pmatrix} = u_{20}$ which already occurs as the vector u_{20} in the original TIFS/10-gadget. The only additional vertex $\mathbf{c} = \begin{pmatrix}0,0,1\end{pmatrix}$ is from the edge connecting u_1 with u_3, as well as u_{20} with u_{22}. thereby "completing" the two cliques/contexts $\{u_1,c,u_3\}$ and $\{u_{20},c,u_{22}\}$. The angle between the two terminal points u_1 and u_{20} of this TITS/11-gadget is $0 < \arccos\frac{1}{\sqrt{1+x^2}} \leq \frac{\pi}{4}$ radians (45°) as $0 < x \leq 1$. This configuration is also a TITS/10-cloud for the 17 pairs $u_1 - \{u_8,u_9,u_{12},u_{13},u_{16},u_{17},u_{22}\}$, $u_6 - u_{22}$, $u_7 - \{u_{12},u_{16},u_{22}\}$, $u_9 - u_{14}$, $u_{10} - u_{22}$, $u_{11} - \{u_{16},u_{22}\}$, $u_{14} - u_{22}$, and $u_{15} - u_{22}$, respectively.

DISCUSSION

It is important to notice that, for *fixed terminal vertices*, depending on the *cloud chosen*, very different classical predictions follow. Indeed, once the terminal vertices are fixed, it is not too difficult to enumerate a quantum cloud which, interpreted classically, predicts and demands *any* kind of input-output behavior. This renders an element of arbitrariness in the interpretation of quantum clouds.

The relevance of this observation lies in the conceivable in-

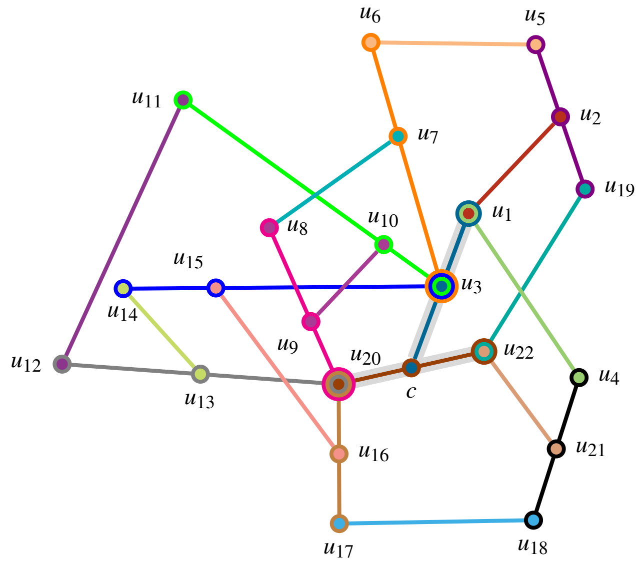

FIG. 7. Orthogonality hypergraph from a proof of Theorem 3 in Ref. [50]. The advantage of this nonfull/TIFS/10-gadget is a straightforward parametric faithful orthogonal representation allowing angles $0 < \angle u_1, u_{22} \leq \frac{\pi}{4}$ radians (45°) of, say, the terminal points u_1 and u_{22}. The corresponding logic including the completed set of 34 vertices in 21 blocks is set representable by partition logics because the supported 89 two-valued states are (color) separable. It is not too difficult to prove (by contradiction) that, say, if both u_1 as well as u_{22} are assumed to be 1, then u_2, u_3, u_4, as well as u_{19}, u_{20} and u_{21} should be 0. Therefore, u_5 and u_{18} would need to be true. As a result, u_6 and u_{17} would need to be false. Hence, u_7 as well as u_{16} would be 1, rendering u_8 and u_{15} to be 0. This would imply u_9 as well as u_{14} to be 1, which in turn would demand u_{10} and u_{13} to be false. Therefore, u_{11} and u_{12} would have to be 1, which yields a complete contradiction even for type-III value assignments. It is also a TITS/11-gadget for the terminal points $u_1 - u_{20}$, constructed by the standard construction depicted in Fig. 6.

terpretation of elementary empirical observations, such as a single particular click in a detector. Suppose a quantum is prepared in a pure state "along" a unit vector $\mathbf{a}$ and, when measured "along" $\mathsf{E}_\mathbf{b} = \mathbf{b}^\dagger\mathbf{b}$, "happens to activate a detector" corresponding to that state $\mathbf{b}$; that is, a detector associated with this latter property clicks. Depending on the quantum cloud considered, the following contradictory claims are justified:

1. if the quantum cloud allows both values then the claim is that there is no determination of the outcome; the event "popped up" from nowhere, *ex nihilo*, or, theologically speaking, has come about by *creatio continua* (cf. Kelly James Clark's God–as–Curler metaphor [76]);

2. in the case of a 10-gadget the system is truly quantum and cannot be classical;

3. in the case of an 11-gadget the system could be classical;

4. in case of a cloud inducing value indefiniteness the

claim can be justified that the system cannot be classical, as no such event (not even its absence) should be recorded. Indeed, relative to the assumptions made, the (non)occurrence of any event at all is in contradiction to the classical predictions.

Conversely, if the experimenter observes no click in a detector associated with the state **b**, then, depending on the quantum cloud considered, the following contradictory claims are justified:

1. as mentioned earlier, if the quantum cloud allows both values then there exists *creatio continua* (currently, this appears to be the orthodox majority position);
2. in the case of a 10-gadget the system could be classical;
3. in case of an 11-gadget the system is truly quantum and cannot be classical;
4. just as mentioned earlier, in case of a cloud inducing value indefiniteness the claim can be justified that the system cannot be classical, as no such event (not even its absence) should be recorded.

As a result, depending on the quantum cloud considered, any (non)occurrence of some single outcome can be published (or rather marketed in venerable scientific journals) as a crucial experiment indicating that the associated system cannot be classical. Likewise, by taking other quantum clouds, any such outcome may be considered to be consistent with classicality: (non)classicality turns out to be *means relative* with respect to the quantum clouds considered. As quantum clouds are configurable for any input-output port setup this is true for any measurement outcome.

The situation turns even more precarious if one considers quantum clouds with a nonunital (and nonseparable) set of two-valued states, such as the ones depicted in Figs. 3 and 4: In the particular faithful orthogonal representation [13, Table 1, p. 102201-7] the vector along $\frac{1}{\sqrt{10}}\left(2\sqrt{2},1,-1\right)$ yields a classical prediction amounting to the *nonoccurrence* of the particular quantum observable. For another example take a cloud introduced by Tkadlec [36, Fig. 2]. It is based on a set of orthogonal vectors communicated to Specker by Schütte [77] and contains 36 vertices in 26 contexts/cliques which allow 6 two-valued states enforcing 8 vertices to be 0. In the particular faithful orthogonal representation of Tkadlec, those correspond to the vectors along $\left(1,0,0\right)$, $\left(0,0,1\right)$, $\left(1,0,1\right)$, $\left(1,0,-1\right)$, $\left(2,0,-1\right)$, $\left(1,0,2\right)$, $\left(-1,0,2\right)$, and $\left(2,0,1\right)$. At the same time the vector $\left(0,1,0\right)$ is forced to be 1. Since without loss of generality, an orthogonal transformation can transform all of these vectors into arbitrary other directions (while maintaining angles between vectors and, in particular, orthogonality) the assumption of such unital configurations and their classical interpretation immediately yields any desired contradiction with any individual measurement outcome.

This arbitrariness could be overcome by some sort of "superselection rule" prioritizing or selecting particular quantum clouds over other ones. However, in the absence of such superselection rules a generalized Jayne's principle, or rather Laplace's principle of indifference, implies that any choice of a particular quantum cloud over other ones amounts to an "epistemic massaging" of empirical data, and their nonoperational, misleading overinterpretation in terms of a speculative ontology [8, 9, 17]; or, to quote Peres [78], *unperformed experiments have no results"*. In contradistinction, it may not be too speculative to hold it for granted that the only operationally justified ontology is the assumption of a single one context or its associated maximal observable.

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